7.10

Let $T: H \to H$ be such that $T(0)=0$ and $\|T(x)-T(y)\| = \|x-y\|$ for all x and y in H. Show that T is a linear isometry from H to H.

Proof: For any $x \in H$, $\|T(x)\| = \|T(x)-T(0)\| = \|x-0\| = \|x\|$.

It remains to show T is linear.

Notice that

$$\begin{aligned}\mathrm{Re}(Tx, Ty) &= \tfrac{1}{2}\left(\|Tx\|^2 + \|Ty\|^2 - \|Tx-Ty\|^2\right) \\ &= \tfrac{1}{2}\left(\|x\|^2 + \|y\|^2 - \|x-y\|^2\right. \\ &= \mathrm{Re}(x,y)\end{aligned}$$

- $\|T(x+y) - T(x) - T(y)\|^2$

$$\begin{aligned}&= (T(x+y)-T(x)-T(y),\ T(x+y)-T(x)-T(y)) \\ &= \|T(x+y)\|^2 + \|T(x)\|^2 + \|T(y)\|^2 - 2\mathrm{Re}(T(x+y), T(x)) \\ &\quad - 2\mathrm{Re}(T(x+y), T(y)) + 2\mathrm{Re}(T(x), T(y)) \\ &= \|x+y\|^2 + \|x\|^2 + \|y\|^2 - 2\mathrm{Re}(x+y, x) \\ &\quad - 2\mathrm{Re}(x+y, y) + 2\mathrm{Re}(x,y) \\ &= (x+y-x-y,\ x+y-x-y) \\ &= \|x+y-x-y\|^2 \\ &= 0\end{aligned}$$

- For any $\alpha \in \mathbb{R}$,

$$\begin{aligned}
&\| T(\alpha x) - \alpha T(x) \|^2 \\
&= (T(\alpha x) - \alpha T(x), T(\alpha x) - \alpha T(x)) \\
&= \|T(\alpha x)\|^2 + \|\alpha T(x)\|^2 - 2\,\mathrm{Re}(T(\alpha x), \alpha T(x)) \\
&= \|\alpha x\|^2 + \alpha^2 \|x\|^2 - 2\,\mathrm{Re}\big(\alpha(\alpha x, x)\big) \\
&= \alpha^2 \|x\|^2 + \alpha^2 \|x\|^2 - 2\alpha^2 \mathrm{Re}(x, x) \\
&= 0.
\end{aligned}$$

Hence, T is $\mathbb{R}$-linear.

T may not be $\mathbb{C}$-linear. Counter-example: $Tz = \bar{z}$.

$\square$

7.11

Let H be a Hilbert space. For each $\phi \in H^*$, let $\nu_\phi \in H$ be the unique element satisfying $\phi(x) = (x, \nu_\phi)$ for all $x \in H$. If $T_0^*: H^* \to H^*$ is the operator adjoint of T, and $T_A^*: H \to H$ is the Hilbert adjoint of T, show that $\nu_{T_0^*\phi} = T_A^* \nu_\phi$ for all $\phi \in H^*$.

Pf: For any x,

$$(x, \nu_{T_0^*\phi}) = T_0^*\phi(x) = \phi(T(x)) = (Tx, \nu_\phi) = (x, T_A^* \nu_\phi).$$

Hence, $\nu_{T_0^*\phi} = T_A^* \nu_\phi$.

$\square$

7.16

Let H be a Hilbert space. If (x_n), (y_n) are sequences in B_H and $\lim_{n\to\infty} (x_n, y_n) = 1$, show that $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Pf: $$\|x_n - y_n\|^2 = 2\|x_n\|^2 + 2\|y_n\|^2 - \|x_n + y_n\|^2$$

$$= \|x_n\|^2 + \|y_n\|^2 - 2\,\mathrm{Re}(x_n, y_n)$$

$$\leq 1 + 1 - 2\,\mathrm{Re}(x_n, y_n)$$

$$\to 0 \quad \text{as } n \to \infty.$$

$\square$